

New Design of RS-GRP LDPC Concatenated Codes for Block Fading Channel

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ABSTRACT

In this paper, we introduce a new code design for block fading (BF) channel by concatenating the Reed-Solomon (RS) codes and generalized root protograph (GRP) low-density parity-check (LDPC) codes in order to achieve the diversity-approaching performance. By decoding iteratively after adapting parity check matrix in the inner RS codes, the diversity-approaching decoding performance can be obtained with low complexity for arbitrary diversity order.

Key Words : Block fading (BF) channel, diversity order, generalized root protograph (GRP) low-density parity-check (LDPC) codes, Reed-Solomon (RS) codes

I. Introduction

Development of channel coding has played a principal role for enhancing reliability in wireless communication systems. Here, modern error correcting codes such as low-density parity-check (LDPC) and polar codes in 5G new radio (NR) are designed for Gaussian channel under the assumption, called the *universality* that they also have a good performance for the other types of channels. However, block fading (BF) channel is an important exception that the universality is not applied generally. Recently, LDPC codes for 5G NR were not analyzed to fully exploit error correction capability when retransmission signal experiences deep fades in the hybrid automatic repeat request (HARQ) system^[1].

A possible alternative is to construct diversity-attaining codes from the existing error correcting codes^[2-4]. The first approach is to use the Reed-Solomon (RS) codes and adaptive decoders^[5].

Also, root LDPC codes and generalized root protograph (GRP) LDPC codes were introduced^[6-8] for the diversity coding. However, the existing classes of the GRP LDPC codes only guarantee the optimal performance with limited code rate in the BF channel or modified HARQ schemes.

In this paper, we propose a novel code design method which concatenates the outer GRP LDPC codes and the inner Reed-Solomon (RS) codes, called the RS-GRP LDPC concatenated codes. Also, a hybrid decoder from the existing adaptive belief propagation (BP) decoder and the sequential decoder is proposed. From the theoretical analysis, the proposed hybrid decoders are shown to have almost-optimal diversity order for the BF channel. Also, numerical analysis shows performance of hybrid decoders with the proposed codes can achieve that of the existing schemes^[9].

This paper is organized as follows, In Section II, preliminaries of the system model with BF channel,

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performance measure, and the existing diversity-attaining coding schemes are introduced. In Section III, a new design of RS-GRP LDPC concatenated codes and their decoders for BF channel are proposed. Also in Sections IV and V, the theoretical and numerical analyses are given for the proposed coding and decoding schemes. Lastly, the paper is concluded in Section VI.

II. Preliminaries

In this section, system model with the BF channel, and the corresponding performance measures and theoretical bounds are introduced.

First, common mathematical notations are summarized as follows. For vector notations, let \mathbf{v} be an n -tuple row vector and v_i be the i -th component of \mathbf{v} . Let $\mathbf{1}$ and $\mathbf{0}$ be all-one and all-zero vectors, respectively. The support set of \mathbf{v} is denoted by $\text{supp}(\mathbf{v}) = \{i; v_i \neq 0, 1 \leq i \leq n\}$ and the Hamming weight of \mathbf{v} is denoted by $\text{wt}(\mathbf{v}) = |\text{supp}(\mathbf{v})|$. Let $[a, b] = \{i; a \leq i \leq b\}$, $a, b \in \mathbb{N}$, $[a] = [1, a]$, and $[0] = \emptyset$ for a set of positive integers \mathbb{N} . Also, denote O_n and I_n as the $n \times n$ zero matrix and the $n \times n$ identity matrix. For $m_r^{(1)} \times m_c^{(1)}$ matrix $M_1 = [m_{i_1, j_1}^{(1)}]$, $i_1 \in [m_r^{(1)}]$, $j_1 \in [m_c^{(1)}]$ and $m_r^{(2)} \times m_c^{(2)}$ matrix $M_2 = [m_{i_2, j_2}^{(2)}]$, $i_2 \in [m_r^{(2)}]$, $j_2 \in [m_c^{(2)}]$, let $M_1 \otimes M_2$ be an $m_r^{(1)} m_r^{(2)} \times m_c^{(1)} m_c^{(2)}$ tensor product matrix of

$$M_1 \otimes M_2 = \begin{pmatrix} m_{1,1}^{(1)} M_2 & \dots & m_{1, m_c^{(1)}}^{(1)} M_2 \\ \dots & \dots & \dots \\ m_{m_r^{(1)}, 1}^{(1)} M_2 & \dots & m_{m_r^{(1)}, m_c^{(1)}}^{(1)} M_2 \end{pmatrix} \quad (1)$$

In the next subsection, we introduce a communication system model with the BF channel.

2.1 System Model with the BF Channel

In this subsection, we consider a nonergodic BF channel with L fading blocks (FBs), where the value of L is small compared to the codeword length and thus, the effect by the BF does not converge to the case of the fast fading. In this paper, we assume that the codes are partitioned into the systematic blocks and the parity

blocks. Also, let l_B be the length of each FB satisfying $N = L \times l_B$ and an information with $K = L_K \times l_B$ bits in the first L_K blocks. Then, the codeword \mathbf{c} with codeword length N is generated by (N, K) linear binary code \mathcal{C} with code rate $R = \frac{K}{N}$. Then, \mathbf{c} is also represented as $\mathbf{c} = (\mathbf{c}_1, \dots, \mathbf{c}_L) = (c_{1,1}, c_{1,2}, \dots, c_{1,l_B}, c_{2,1}, c_{2,2}, \dots, c_{L,1}, \dots, c_{L,l_B}) \in \mathbb{F}_2^N$, where \mathbb{F}_2 denotes the binary field, \mathbf{c}_i denotes a vector of coded bits $c_{i,j}$, $j \in [l_B]$ in the i -th block and $l_B \gg L$. Here, the coded bits consist of messages in $\mathbf{c}_1, \dots, \mathbf{c}_{L_K}$ and parities in $\mathbf{c}_{L_K+1}, \dots, \mathbf{c}_L$. Here, the minimum block-wise Hamming weight d_B is given as

$$d_B := \min_{\mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}} \left(\sum_{i \in [L]} u(\text{wt}(\mathbf{c}_i)) \right), \quad (2)$$

where $u(x)$ is a step function which returns 1 if $x > 0$ and returns 0, otherwise. By binary phase shift keying (BPSK) modulation of binary constellation $\mathcal{X} = \{-1, 1\}$, \mathbf{c} is converted to $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_L) \in \mathcal{X}^N$ with $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,l_B})$. After transmitter (Tx) constitutes a signal, the signal power is expressed as $y_{i,j} = \alpha_i x_{i,j} + z_{i,j}$, $i \in [L]$, $j \in [l_B]$, where α_i follows a Rayleigh distribution with zero mean and the second moment $E[\alpha_i^2] = 1$ and $z_{i,j}$ is a real additive Gaussian noise with variance $2\sigma^2$. In the receiver (Rx) side, the receiver corrects the erroneous codewords using the soft-decisioned iterative decoder, which is based on message passing algorithm in the bipartite graph between variable nodes (VNs) and check nodes (CNs) from the parity check matrix (PCM). Also, the iterative decoder exploits the log-likelihood ratios (LLRs) of codeword bits with the perfect channel state information such as channel coefficients and noise variance. From the received bit $y_{i,j}$ the channel LLR Λ_i^{ch} is derived and each element $\Lambda_{l_B(i-1)+j}^{ch}$ of the iterative decoder is written as

$$\Lambda_{l_B(i-1)+j}^{ch} = \frac{2\alpha_i y_{i,j}}{\sigma^2} = \frac{2}{\sigma^2} (\alpha_i^2 x_{i,j} + \alpha_i z_{i,j}), \quad (3)$$

where the subscript of $\Lambda_{l_B(i-1)+j}^{ch}$ denotes the index of VNs in the bipartite graph.

For the decoding procedure, iterative sum-product algorithm (SPA) or min-sum algorithm (MSA) is used, which consists of variable node update (VNU) and check node update (CNU) procedure. In order to analyze the BF channel, we use the assumption that the BF channel with an L -tuple fading vector $\mathbf{v}_\alpha = (\alpha_1, \dots, \alpha_L)$ is considered as a set of independent channel realization with different signal-to-noise ratio (SNR)^[10].

2.2 Performance Measures and Theoretical Bound

In this paper, frame error rate (FER) and diversity order are considered as performance measures. First, let $P_F^L(\bar{\gamma})$ be the FER of a coded transmission with a code rate R over L FBs of size I_B at SNR of $\bar{\gamma}$, where FER indicates the rate of failure to recover the information message $(\mathbf{c}_1, \dots, \mathbf{c}_{L_K})$.

Also, an outage probability is defined as a lower bound of the FER over the code rate R and L FBs with coefficients of random variable $\mathbf{v}_\alpha = \{\alpha_1, \dots, \alpha_L\}$ as

$$P_{out}^L(R, \bar{\gamma}) = P_{V_\alpha} \left[\frac{\sum_{i \in [L]} I_{AWGN}(\alpha_i^2 \bar{\gamma})}{L} \leq R \right], \text{ where } I_{AWGN}(\cdot)$$

denotes the instantaneous modulation constrained input-output mutual information (MI) of an additive white Gaussian noise (AWGN) channel with BPSK modulation^[9]. By sampling the large numbers with L realizations of the fading coefficients, approximated values on the outage probability $P_{out}^L(R, \bar{\gamma})$ can be obtained. However, it requires a huge number of iterations to get an accurate point with the large FBs L and lower FER. Instead, we use the following theoretical bound with an optimal slope.

Note that the MI of the i -th FB with a fading coefficient α_i is given as

$$I_{AWGN}(\alpha_i^2 \bar{\gamma}) = 1 - E_{X,Z} \left\{ \log_2 \left(\sum_{x \in \mathcal{X}} e^{-|\sqrt{\bar{\gamma}}(X-x)+Z|^2 + |Z|^2} \right) \right\} \quad (4)$$

$$\leq \min\{\log_2(1 + \gamma_i), 1\},$$

where X is a random variable uniformly distributed over BPSK modulation and $Z \sim \mathcal{N}(0, \sigma^2)$. By using the upper bound in (4), the lower bound of the outage

probability is derived as [8]

$$P_{out}^L(R, \bar{\gamma}) \geq \sum_{t=0}^L F_{A_t}(LR-t) \binom{L}{t} (p\bar{\gamma})^t (1-p\bar{\gamma})^{L-t}, \quad (5)$$

where $P\bar{\gamma} = \Gamma(1, 1/\bar{\gamma})$ with the incomplete gamma function $\Gamma(a, b)$ and $F_{A_t}(t)$ is a cumulative distribution function of $A_t = \log_2(1 + \gamma_i)$. Note that the bound in (5) is useful to estimate the outage probability for arbitrary L and R .

For $I \in [L]$, let $\alpha_{i:L}$ and $\gamma_{i:L}$ be the i -th ordered statistics of α_i and γ_i satisfying $\alpha_{1:L} \geq \alpha_{2:L} \geq \dots \geq \alpha_{L:L}$ and $\gamma_{1:L} \geq \gamma_{2:L} \geq \dots \geq \gamma_{L:L}$ among L realizations of α and γ , respectively^[11]. Here, we denote a map $\pi(i) = I$ from the FB index i to the index of power order, $I \in \{\gamma_{1:L}, \gamma_{2:L}, \dots, \gamma_{i:L}, \dots, \gamma_{L:L}\}$ and its inverse map $\pi^{-1}(I) = i$.

As a performance measure, a concept of the diversity order is defined as follows.

Definition 1 (Diversity order d_c).

$$d_c := - \lim_{\bar{\gamma} \rightarrow +\infty} \frac{\log P_F^L(\bar{\gamma})}{\log \bar{\gamma}}.$$

Although the diversity order is defined as a slope of the coded FER in the asymptotic SNR, diversity order also determines the FER in the finite SNR, where the slope of FER is observed to be constant. Then, the well-known bound of the diversity order is introduced as follows.

Proposition 1 (Singleton-like bound). *For a code with rate R , diversity order d_c and blockwise Hamming distance d_B , we have*

$$d_c \leq d_B \leq 1 + \lfloor L(1-R) \rfloor. \quad (6)$$

If $d_c = d_B = 1 + \lfloor L(1-R) \rfloor$ from (6), it is said that the code has the diversity-achieving performance if two equalities of $d_c = d_B$ and $d_B = 1 + \lfloor L(1-R) \rfloor$ are satisfied, which should be proven respectively. However, the only few codes are known to be diversity-achieving satisfying the strict condition. Instead,

the code is said to be diversity-approaching if $\frac{\log P_E^L(\tilde{\gamma})}{\log \tilde{\gamma}} \approx -(1 + \lfloor L(1 - R) \rfloor)$ for a finite $\tilde{\gamma}$, which is a relaxed condition compared to the diversity-achieving code, but it can be shown by the probabilistic approach or comparison of the slope of FER with the lower outage probability in the finite-length analysis^[7]. In the next subsection, the existing diversity-approaching RS codes and GRP-LDPC coded HARQ-IR for BF channel are introduced.

2.3 Existing Diversity-Approaching RS Codes and GRP-LDPC Coded HARQ-IR System for BF channel

In this subsection, two existing diversity-approaching schemes for general L are discussed. First, we introduce the existing diversity-approaching RS coding schemes for a comparison^[9], where the RS code is one of the well-known maximum distance separable (MDS) codes. [9] proposed an adaptive BP decoder which incorporates order statistics decoding (OSD) and soft-decisioned BP decoding. From the system model in Section II-A, we assume that the RS code with L codeword symbols and L_K message symbols is used for the error correcting codes over the finite field \mathbb{F}_{2^m} with $m \geq \lceil \log_2(L + 1) \rceil$. Also, we define a companion matrix T^i denoted by

$$T^i = : [[\alpha^i, [\alpha^{i+1}], \dots, [\alpha^{i+m-1}]]] \in \mathbb{F}_2[T] \subseteq \mathbb{F}_2^{m \times m},$$

where $[\alpha^i]$ is denoted as an $m \times 1$ column vector from the additive m -tuple representation of a finite field with a basis $\{1, \alpha, \dots, \alpha^{m-1}\}$ and $\mathbb{F}_2[T] = \{T^0, T^1, \dots, T^{2^m-2}\}$. Let \mathcal{S} be the field-isomorphic map from $\{\alpha^0, \alpha^1, \dots, \alpha^{2^m-2}\} \subseteq \mathbb{F}_{2^m}$ to $\mathbb{F}_2[T]$. Then, a PCM of RS codes is represented as follows.

Construction 1 (Binary representation of PCM for RS codes). For $l_B = m$, the binary form of PCMs for $(N, K) = (mL, mL_K)$ RS code HRS is given as

$$H_{RS} = \begin{pmatrix} T^0 & T^0 & \dots & T^0 \\ T^0 & T^1 & \dots & T^{L-L_K-1} \\ \dots & \dots & \dots & \dots \\ T^0 & T^{L_K-1} & \dots & T^{(L-L_K-1)(L-1)} \end{pmatrix}. \quad (7)$$

In order to obtain the diversity order in the RS codes in the BF channel, a modified BP decoder called adaptive BP decoder is proposed by adding the procedure of adapting the PCM of (7) as Definition 2.

Definition 2 (Adaptive BP decoder for RS codes). Adaptive BP decoder for the RS codes is operated as in the following three phases:

1. After the Rx obtains the channel LLR as in (3), select $L - L_K$ smallest symbol indices of VNs in terms of the absolute values of channel LLR of the RS codes, whose set is denoted by \mathcal{B} . Then, the corresponding submatrix of their PCM $D_{RS}^{(\mathcal{B})}$ is an $m(L - L_K) \times m(L - L_K)$ binary matrix whose bit columns are located in the VN symbol index set \mathcal{B} from H_{RS} .
2. Construct an $(N - K) \times N$ PCM H'_{RS} by computing $H'_{RS} = (D_{RS}^{(\mathcal{B})})^{-1} H_{RS}$.
3. Decode the received word using the iterative algorithm such as SPA or MSA using the modified PCM of RS codes, H'_{RS} .

For long l_B in the BF channel, the maximum diversity can be achieved by using $\frac{l_B}{N}$ RS codes, called block-wise RS code. In this procedure, the adaptive BP decoder use matrix inversion operation of $(D_{RS}^{(\mathcal{B})})^{-1}$ and BP operation. Note that that the complexity of the matrix inversion is represented by $\mathcal{O}(\frac{N}{m})^3$ for Gauss-Jordan method.

On the other hands, the modified GRP LDPC coded H-AHQ-IR system can approach the maximum diversity for given rate. For this, GRP LDPC codes are defined as in Construction 2.

Construction 2 (GRP LDPC codes). For the number of FBs L in the initial transmission, let M be a $bL \times bL$ lower triangular matrix with nonnegative integer entries and all-one diagonal elements, which is equally row-partitioned with row size b as $M = [M_1^T, M_2^T, \dots, M_L^T]^T$. For $\forall l_B$, an $l_B \times l_B L$ the GRP LDPC code is represented by the PCM HO lifting

from the $bL \times bL^2$ base matrix $B_{GRP} = [B_1, B_2, \dots, B_L]$ with $B_i = [M_{i+1}^\top, M_{i+2}^\top, \dots, M_L^\top, M_1^\top, \dots, M_i^\top]^\top$, for $i \in [L]$ and $B_L = [M_1^\top, M_2^\top, \dots, M_L^\top]^\top$.

In addition, the corresponding HARQ-IR system with maximum number of FB L is combined with a sequential decoder as follows.

Definition 3 (Sequential decoder for GRP LDPC-coded HARQ-IR system). *Sequential decoder operates the following three phases:*

- i) *In the first step, the first bit in c_{L_K+1} is generated from the K message bits and the CNs which are connected with the VN with weight one as in the first row of B_{GRP} . Also, The other bits in c_{L_K+1} are sequentially generated from K message bits and generated parity, and the CNs which are connected to the VN of weight one and the VNs in the protograph.*
- ii) *If the decoding of an initial transmission is failed, the receiver transmits additional feedback signal including the index with fading block of the worst channel coefficient $\forall L:L$ as well as NAK. Then, the Tx changes the configuration between CNs and VNs by puncturing the code bits of the worst FB and extending a new IR block.*
- iii) *After LDPC decoding is performed under the modified configuration in ii), the remaining process depends on the decoding status.*
 - *If the initial decoding is successful, the punctured bits can be recovered using the configuration of the first transmission, where the punctured bits are recovered by the encoding operation in i).*
 - *If LDPC decoding fails, the Tx and Rx repeats the same procedure of ii) until the new IR makes the codeword recovery successful or the number of retransmissions τ reaches the maximum value $\tau_{max} = L - L_K$.*

However, sequential decoding can be applied only in the HARQ-IR system with an additional feedback

signal. Also, processing latency and decoding complexity are increased by the number of retransmissions.

In order to remove the drawbacks of the existing schemes, a new design and a hybrid decoder for diversity-approaching RS-GRP LDPC concatenated codes are proposed in the next section.

III. New Design of RS-GRP LDPC Concatenated Codes and Hybrid Decoders

In this section, new code design of the RS-GRP LDPC concatenated codes and hybrid decoder are proposed. First, we propose a new code design in the following subsection.

3.1 New Design of RS-GRP LDPC concatenated codes

In this subsection, a new design of RS-GRP LDPC concatenated codes is introduced as in the following construction.

Construction 3 (RS-GRP LDPC Concatenated Codes). *For $N = Ll_B$ and $K = L_K l_B$, (N, K) RS-GRP LDPC concatenated code is represented by the PCM*

$$H = \begin{bmatrix} H_O & O \\ & H_I \end{bmatrix}, \text{ where two submatrices of PCMs } H_O \text{ and } H_I \text{ are the PCMs of outer } ((L_K + 1)l_B, L_K l_B) \text{ GRP LDPC codes and the inner } (Ll_B, (L_K + 1)l_B) \text{ RS codes.}$$

First, PCM of GRP LDPC codes H_O is constructed as in Construction 2 with $L_K + 1$ FBs. Also, a PCM of the inner RS code H_I with symbol size m can constructed by $H_I = I_{l_B} \otimes H_{IRS}$, where H_{IRS} is the PCM of an $(N, K) = (mL, n(L_K + 1))$ RS code with PCM of (7).

Note that the PCM of H_{IRS} is a sparse matrix $n(L - L_K - 1) \times mL$ with average CN degree $\frac{Lm}{2}$, which has an advantage of good performance under the iterative decoding. For a FB index set W , let \mathcal{B}_W and \mathcal{B}'_W be bit index sets of

$$\mathcal{B}_W = \bigcup_{i \in W} \bigcup_{j \in [l_B]} \{il_B + (j-1)\}, \mathcal{B}'_W = \bigcup_{i \in W} \bigcup_{j \in [m]} \{im + (j-1)\}. \tag{8}$$

Then, the encoding of the proposed code generates the parity bits from the outer GRP LDPC codes in \mathbf{c}_{L_K+1} and from the inner RS codes in $\{\mathbf{c}_{L_K+2}, \dots, \mathbf{c}_L\}$ subsequently. Here, the encoding can be conducted with lower complexity using the structural characteristics of the PCM for GRP LDPC codes H_O and block-wise RS codes with PCM $H'_I = \left(D_I^{(\mathcal{B}_F)}\right)^{-1} H_{IRS}$, where $F = [L_K + 1, L]$ and $\mathcal{B}_F = [(L_K + 1)l_B + 1, Ll_B]$. Here, $\left(D_I^{(\mathcal{B}_F)}\right)^{-1}$ can be computed with lower complexity by tensor decomposition $\left(D_I^{(\mathcal{B}_F)}\right)^{-1} = I_{l_B} \otimes \left(D_{IRS}^{(\mathcal{B}'_F)}\right)^{-1}$, where $D_{IRS}^{(\mathcal{B}'_F)}$ is an $(L - L_K - 1)m \times (L - L_K - 1)m$ sub-matrix of H_{IRS} with columns in the index set $\mathcal{B}'_F = [(L_K + 1)m + 1, Lm]$.

In the next subsection, the hybrid decoding algorithms for the proposed codes will be discussed.

3.2 Hybrid Decoding Algorithms for RS-GRP LDPC Concatenated Codes

In the next subsection, a new hybrid decoding algorithms for the proposed codes is introduced. In order to take advantages of two existing decoders, their hybrid decoders for the proposed RS-GRP LDPC codes are newly designed. Here, the proposed hybrid decoder follows the procedure of adaptive BP decoders in the inner RS codes and sequential decoders in the outer GRP LDPC codes. First, the Rx is assumed to know the values of BF coefficients and thus, a set of bit indices of the worst $L - L_K - 1$ fading coefficients W and the set of corresponding bit indices in the FBs of \mathcal{B}_W and \mathcal{B}'_W can be obtained as in (8).

Definition 4 (Hybrid decoders). *Classes of hybrid decoders are divided into the following two types, hybrid-I and hybrid-II decoders, each of which consist of the following three phases:*

- i) Construct a set of FB index set W with size

$(L - L_K - 1)$, a set of $(L - L_K - 1)l_B$ bit indices \mathcal{B}_W and a corresponding submatrix $D_I^{(\mathcal{B}_W)}$ of PCM based on the magnitude of block fading vector α . Fig. 1 shows the PCM of the RS-GRP LDPC concatenated codes with the set of FB index set \mathcal{B}_W with the minimum index two.

- ii) Obtain the submatrix H'_I of PCM from H_I in the inner block-wise RS code by the PCM $H'_I = \left(D_I^{(\mathcal{B}_W)}\right)^{-1} H_{IRS}$. Then, the PCM is modified using H'_I and H_O . Fig. 2 illustrates the modified PCM of the RS-GRP LDPC concatenated codes with the set of FB index set \mathcal{B}_W , where the identity submatrices I are placed on the column index \mathcal{B}_W
- iii) Using the modified PCM, the BP decoding operation is performed. In this step, the BP decoder can selectively use the different sub-matrix in each iteration or whole PCM for the decoding. Specifically, it is divided into the following two types:

- For the case of hybrid-I decoder, the decoders substitute the received channel LLRs in the worst $L - L_K - 1$ FBs into the extrinsic LLRs using H'_I in inner block-wise RS codes.

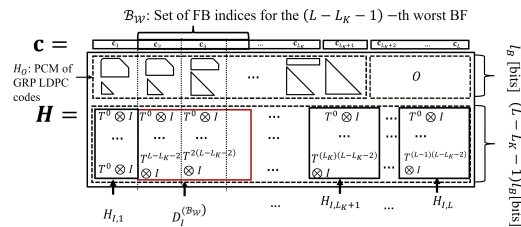


Fig. 1. PCM of the RS-GRP LDPC concatenated codes with the set of FB index set \mathcal{B}_W

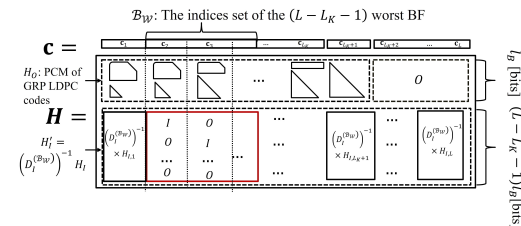


Fig. 2. Modified PCM of the RS-GRP LDPC concatenated codes with the set of FB index set \mathcal{B}_W

Then, the decoder corrects the codeword using an iterative BP algorithm with the configuration of H_O , PCM of the outer GRP LDPC codes.

- For the case of hybrid-II decoder, BP decoding is performed from the whole configuration of $H' = \begin{bmatrix} H_O & O \\ & H'_I \end{bmatrix}$.

For the first phase, the procedure of adapting the PCM is equivalent to computing $H'_I = (D_I^{(\mathcal{B}_W)})^{-1} H_I$. Similar to the encoding procedure, the complexity of matrix multiplication and inversion is low by computing the small-sized submatrix $D_I^{(\mathcal{B}_W)}$ in $(D_{\mathcal{B}_W})^{-1} = I_{\frac{L}{m}} \otimes (D_{IRS}^{(\mathcal{B}_W)})^{-1}$ as in the encoding procedure from \mathcal{B}_W .

For the third phase of the hybrid-I type decoder, the channel LLR in \mathcal{B}_W , Λ_i^{ch} , $i \in \mathcal{B}_W$ is substituted by the extrinsic LLR Λ_i^{sub} with VN indices of $[M] \setminus \mathcal{B}_W$. Then, the outer GRP LDPC codes decode the sub-codeword $(c_1, \dots, c_{L-L_K+1})$ using the partially substituted channel LLRs. For SPA, we have

$$\Lambda_i^{sub} = 2 \tanh^{-1} \left(\prod_{j \in C_i \setminus \{i\}} \tanh \left(\frac{1}{2} \Lambda_j^{ch} \right) \right) \text{ for SPA, } i \in W. \quad (9)$$

For MSA, we have

$$\Lambda_i^{sub} = \prod_{j \in C_i \setminus \{i\}} \{ \text{sign}(\Lambda_j^{ch}) \} \min_{j \in C_i \setminus \{i\}} \{ |\Lambda_j^{ch}| \} \text{ for MSA, } i \in W, \quad (10)$$

where C_i is the set of VNs connected to the CNs with the connection of the i -th VN for $i \in W$ in the configuration H'_I .

Note that the configuration of the modified submatrix H'_I is designed to achieve a diversity-approaching FER performance with the minimum diversity order $L - L_K$ by SPA and MSA for any parameters of L and R . In the next section, the corresponding theoretical analysis will be discussed.

IV. Theoretical Analysis of the Proposed RS-GRP LDPC Concatenated Codes

In this section, theoretical proof for the diversity order of the proposed RS-GRP LDPC concatenated codes is introduced. Here, suppose that for all-zero code-word, MSA with the hybrid-I decoder is used. Note that the subscript of the channel LLR stands only for the FB index but this notation is still sufficient to show message passing of LLR by the block-wise RS codes after the first stage of the hybrid-I decoding. Then, we can show the diversity-guaranteeing property as in the following theorem.

Theorem 1. *The diversity order of the RS-GRP LDPC concatenated codes with the hybrid-I decoder is at least $L - L_K$.*

Proof. In order to show that the coded performance achieves at least diversity order $L - L_K$ in an asymptotic sense, we track the behavior of a priori LLRs of the hybrid-I decoding for the inner RS codes and outer GRP LDPC codes, respectively.

After the first stage of hybrid-I decoder, the substituted LLR is lower bounded as follows.

Lemma 1. *After the first stage of MSA in the hybrid-I decoder, there exist finite values of I and i , where a substituted LLR of i -th VNs is lower bounded by*

$$\alpha_{L_K+1:L}^2 - \alpha_{L_K+1:L} \max_{i \in [I]} |z_i| \leq \Lambda_i^{sub}. \quad (11)$$

Proof. Recall that Λ_i^{sub} can be represented as (10). For the Tanner graph representation of H'_I , each CN is connected with a VN with index \mathcal{B}_W and the other VNs with $[M] \setminus \mathcal{B}_W$. If $i \in \mathcal{B}_W$, If $i \in \mathcal{B}_W$ in the substitution procedure (10), $C_i \setminus \{i\} \in [M] \setminus \mathcal{B}_W$. As a minimum of $\min_{j \in C_i \setminus \{i\}} \{ |\Lambda_j^{ch}| \}$ with (3), the channel LLR with fading coefficient $\alpha_{L_K+1:L}$ is selected. In other words, the modified PCM H'_I guarantees that Λ_i^{sub} should be represented by (11), where Λ_i^{ch} with BF coefficients $\alpha_{1:L}, \dots, \alpha_{L_K+1:L}$, are connected. After min-sum decoding of the outer GRP LDPC codes,

Λ_i^{sub} follows (11) because MSA only modifies the size of I in the variable node update (VNU) process as in Theorem 4 in [6] and thus, the values of I and i should be finite. \square

In order to understand the LLR behavior by iterative decoding of the outer GRP LDPC codes, we will revisit the existing definition as follows.

Definition 5. Let Δ_i be the spacing defined as $\Delta_L = \gamma_{L,L}$ for $i = L$ and $\Delta_i = \gamma_{i,L} - \gamma_{i+1,L}$, otherwise. Then, moment generating function (MGF) of $\gamma_\Delta = \sum_{l \in [L]} a_l \Delta_l$ can be written as

$$\mathcal{M}_{\gamma_\Delta}(s) = \left(1 - \frac{\alpha_l \bar{\gamma} s}{l}\right)^{-1}. \quad (12)$$

In addition, iterative decoding of the outer GRP LDPC codes consists of VNU and check node update (CNU). After VNU, a posteriori LLR Λ_i^P of i -th VN, which is decoded LLR from MSA is represented as

$$\Lambda_i^P = \Lambda_i^{ch} + \Lambda_i^{ext},$$

where Λ_i^{ch} and Λ_i^{ext} are the channel and extrinsic LLRs of the i -th VN.

Using Lemma 1 and Definition 5, the upper bound of diversity order of the proposed codes can be derived as follows.

Note that the proof can be easily done by the similar procedure as in Theorem 1 in [6]. In the first stage of the hybrid-I decoder, the channel LLRs with the $L - L_K - 1$ worst FBs in the inner RS code are substituted. To this ends, the channel LLRs are divided into two subcases: substituted channel LLRs of the index set \mathcal{S} and non-substituted channel LLRs of the index set $\bar{\mathcal{S}}$.

After decoding inner block-wise RS code, the channel LLRs of the outer GRP LDPC codes with the first L FBs are reordered by $\Lambda_{i:L}^{ch}$ satisfying $\Lambda_{1:L}^{ch} \geq \Lambda_{2:L}^{ch} \geq \dots \geq \Lambda_{L_K+1:L}^{ch}$, where the channel LLR values with index $i \in [L \setminus \bar{\mathcal{S}}]$ are from non-substituted channel LLRs, and from substituted channel LLRs, otherwise.

Then, suppose that both Λ_i^{ch} and Λ_i^{ext} are modified from the substituted LLRs, where the substituted LLRs have the index j and the smallest magnitude among the other channel LLRs. Then, the error probability by a posteriori LLR Λ_i^P is bounded as

$$\begin{aligned} Pr(\Lambda_i^P < 0) &\leq Pr(\Lambda_{L_K+1:L}^{ch} + \Lambda_{L_K+1:L}^{ext} < 0) \\ &= Pr(\Lambda_i^{sub} + \Lambda_j^{sub} < 0) \leq 1 - Pr(\Lambda_i^{sub} > 0)^2 \\ &= 1 - (1 - Pr(\Lambda_i^{sub} < 0))^2 \approx 2Pr(\Lambda_i^{sub} < 0). \end{aligned}$$

In other words, the condition of $\{\Lambda_i^{sub} < 0\}$ is considered as an upper bound for the error rate of a posteriori LLR, whose probability can be expressed as

$$Pr\left(\frac{2}{\sigma^2}(\alpha_{L_K+1:L}^2 < \alpha_{L_K+1:L} \max_{i \in [L]} |z_i|)\right) \quad (13)$$

by (11). Using Q-function, (13) is rewritten as

$$\begin{aligned} d_c &\geq - \lim_{\bar{\gamma} \rightarrow +\infty} \frac{\log P_F(\bar{\gamma})}{\log \bar{\gamma}} \\ &\geq - \lim_{\bar{\gamma} \rightarrow +\infty} \frac{\log(1 - (1 - E_{\gamma_{L_K+1:L}} Q(\sqrt{\gamma_{L_K+1:L}}))^l)}{\log \bar{\gamma}} \\ &\geq - \lim_{\bar{\gamma} \rightarrow +\infty} \frac{\log I \times E_{\gamma_{L_K+1:L}} Q(\sqrt{\gamma_{L_K+1:L}})}{\log \bar{\gamma}} \\ &\geq - \lim_{\bar{\gamma} \rightarrow +\infty} \frac{\log \mathcal{M}_{\gamma_{L_K+1:L}}(-\frac{1}{2})}{\log \bar{\gamma}} = d'_c, \end{aligned} \quad (14)$$

where the inequality of the each line in (14) is derived by using $P(\max |z| < T) = 1 - (1 - Q)^l$, $Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$ and $E_{\gamma_{L_K+1:L}}(e^{\gamma_{L_K+1:L}/2}) = \mathcal{M}_{\gamma_{L_K+1:L}}(-\frac{1}{2})$, and Definition 5, respectively. Thus, the theorem is proved. If either channel or extrinsic LLR of the i -th FB is from non-substituted LLR, the proof can be done as in the same procedure for $I = 1$. \square

Recall that the guaranteed diversity order by Theorem 1 is sub-optimal by (6). However, for the finite-length cases, numerical results of FER performance behavior show that the proposed codes have diversity-approaching performances, which are discussed in the next section.

V. Numerical Analysis of Performance of the Proposed Coding and Decoding Schemes

From the numerical analysis, we compare the FER performance of the proposed RS-GRP LDPC concatenated codes with that of the $(L, L_{\kappa} = 4)$ block-wise RS codes in Construction 1 and the existing adaptive BP decoding algorithm in [9]. Here, the adaptive BP decoding has the largest complexity because the stages of adapting the PCM and iterative decoding in the existing adaptive BP decoder should be performed with larger matrix size and higher degree than those in the proposed decoding algorithm.

For the design of the proposed RS-GRP LDPC concatenated codes, the base matrix of the outer GRP LDPC codes B_O is firstly optimized as

$$B_O = \begin{pmatrix} \begin{array}{c|c|c|c|c} 11031 & 12110 & 00100 & 01000 & 10000 \\ 10000 & 11031 & 12110 & 00100 & 01000 \\ 01000 & 10000 & 11031 & 11210 & 00100 \\ 00100 & 01000 & 10000 & 11031 & 12110 \\ 12110 & 00100 & 01000 & 10000 & 11031 \end{array} \end{pmatrix} \quad (15)$$

using the PEXIT algorithm as in [6], [7], with the constraints of maximum values of elements in B_O as 3. Then, the base matrix is converted into the parity check matrix by the lifting procedure with lifting size 840 and the girth larger than six, which generates the $(N, K) = (21000, 16800)$ outer GRP LDPC codes. For the range of $6 \leq L \leq 8$, the inner block-wise RS codes are defined from the $(4L, 20)$ block-wise RS codes with primitive root α as a root of $p(x) = x^4 + x + 1$. In addition, $(4L, 16)$ block-wise RS codes with adaptive BP decoder and GRP-LDPC coded HARQ-IR system with base matrix B_O are also analyzed for the comparison.

Fig. 3 shows the FER performance of the proposed RS-GRP LDPC concatenated codes and the existing block-wise RS codes and GRP LDPC coded HARQ-IR system. For performance of the proposed codes with the hybrid-I and hybrid-II decoders, their FER slopes are nearly identical to the lower outage

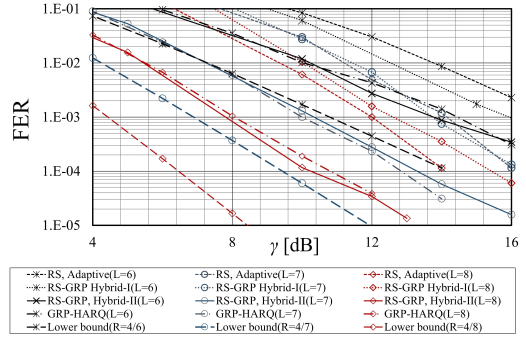


Fig. 3. FER performance of the proposed RS-GRP LDPC concatenated codes and the existing block-wise RS codes and GRP-LDPC coded HARQ-IR system.

bound for all values of L , which means that the proposed scheme can achieve the optimal diversity order as in the outage bound in the low $\tilde{\gamma}$. Note that our experiment is conducted under the coding $R = \frac{4}{L}$ with the diversity order $d_c \approx L - L_{\kappa} + 1$ according to the definition of the diversity-approaching codes. In Fig. 3, steeper slopes in the FER curves are observed for the codes with higher L by the diversity orders d_c .

As another consideration of the performance analysis, coding gain of the proposed scheme can be checked by comparison with the existing two schemes of the adaptive decoder of RS codes and the sequential decoder of GRP LDPC codes with the HARQ-IR.

When compared with existing RS codes and adaptive decoder, the proposed hybrid-I and II decoders are observed to have large coding gains upto 2 [dB] and 4 [dB] compared to RS adaptive decoders at $L = 6$ in the FER region of 10^{-3} , respectively. On the other hands, the proposed codes with hybrid-II decoder and the existing sequential decoder of GRP-LDPC codes has lower FER compared to the proposed codes with hybrid-I decoder. In addition, the FER performances of proposed codes with hybrid-II decoder and sequential decoder are similar, which means the the proposed scheme can achieve the performance of the existing diversity-approaching schemes for larger L . On the other hands, the proposed codes with hybrid-II decoder has an advantage in the implementation aspect because additional feedback and HARQ-IR system are not necessarily required.

VI. Conclusion

In this paper, we proposed a new class of RS-GRP LDPC concatenated codes, which achieves near-optimal diversity order comparable to the existing codes. Nevertheless, the coded performance can be further improved by the detailed optimization process because the proposed code design only considers to have a good diversity order in an asymptotic sense. Therefore, the optimized decoding for the finite SNR may derive an improvement which reduces the gap between the coded performance and the theoretical lower bound.

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